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Some Basic Properties of Weakly Classical Primary and Classical Primary Subsemimodules in Semimodules

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ABSTRACT— *In this paper, we study classical primary and weakly classical primary subsemimodules in semimodules over a semirings. Some characterizations of classical primary and weakly classical primary subsemimodules are obtained. Moreover, we investigate relationships classical primary and weakly classical primary subsemimodules in semimodules over a semirings. Finally, we obtain necessary and sufficient conditions of a weakly classical primary to be a classical primary.*

Keywords— semimodule, classical primary subsemimodule, weakly classical primary subsemimodule, k -subsemimodule, primary ideal.

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1. INTRODUCTION

Throughout this paper a semiring will be defined as follows: A semiring is a set R together with two binary operations called addition " + " and multiplication " $\cdot$ " such that $(R,+)$ is a commutative semigroup and $(R,\cdot)$ is semigroup; connecting the two algebraic structures are the distributive laws : $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$ for all $a, b, c \in R.$ A subset A of a semiring R is called an ideal of R if for $a, b \in A$ and $r \in R,$ $a + b \in A, ar \in A$ and $ra \in A.$ A proper ideal P of R is called a primary ideal if $ab \in P,$ where $a, b \in R,$ implies that either $a \in P$ or $b^n \in P,$ for some positive integer $n.$ P is said to be quasi-primary if for all $a, b \in R$, $ab \in P$ implies that either $a^n \in P$ or $b^n \in P,$ for some positive integer $n.$ Clearly every primary is a quasi-primary. A semimodule M over a semiring R is a commutative monoid M, together with a function $R \times M \to M$, defined by $(r, m)rm$ such that:

1. $r(m+n) = rm + rn$
2. $(r+s)m = rm + sm$
3. $(rs)m = r(sm)$
4. $1m = m$

for all $m, n \in M$ and $r, s \in R.$ Clearly every ring is a semiring and hence every module over a ring R is a left semimodule over a semiring $R.$ A nonempty subset N of a R -semimodule M is called subsemimodule of M if N is closed under addition and closed under scalar multiplication. A proper subsemimodule N of an R -semimodule M is said to be primary if $rm \in N$, $r \in R, m \in M$, then either $m \in N$ or $r^n M \subseteq N$, for some positive integer $n.$

J. Saffar Ardabili, S. Motmaen and A. Yousefian Darani in (2011) defined a different class of subsemimodules and called it classical prime. A proper subsemimodule N of M is said to be classical prime when for $a, b \in R$ and $m \in M$, $abm \in N$ implies that $am \in N$ or $bm \in N.$ In this paper we introduce the concept of classical primary subsemimodules of a semimodule M, and study some basic properties of this class of subsemimodules. Moreover, we investigate relationships between classical primary and weakly classical primary of subsemimodules in M.

2. BASIC PROPERTIES OF SUBSEMIMOMDULES

In this section we refer to [1, 5] for some elementary aspects and quote few theorem and lemmas which are essential to step up this study. For more details we refer to the papers in the references.

Definition 2.1. [5] Let M be an R-semimodule and N be a proper subsemimodule of M. An associated ideal of N is defined as $(N : M) = \{a \in R : aM \subseteq N\}$.

Lemma 2.2. [5] Let M be an R-semimodule and N be a proper subsemimodule of M. If N is a subtractive subsemimodule of M, and let $m \in M$. Then the following hold:

1. $(N : M)$ is a subtractive ideal of R.
2. $(0 : M)$ and $(N : m)$ are subtractive ideals of R.

Lemma 2.3. [1] Let N and K be subsemimodules of a semimodule M over a semiring R with $N \subseteq K$. Then K / N is a subsemimodule of M / N.

Definition 2.4. Let N be any subsemimodule of an R-semimodule M. For $a \in R$ and an ideal I of R, the sets $[N : a]$ and $[N : I]$ are defined by

1. $[N : a] = \{m \in M : am \in N\}$
2. $[N : I] = \{m \in M : Im \subseteq N\}$
3. $\sqrt{[N : a]} = \{a^n m \in N : n \in\}$

Remark. Let N be any subsemimodule of an R-semimodule M and let I be an ideal of R, $a \in R$. Then

1. $[N : a] \neq \varnothing$ and $[N : I] \neq \varnothing$
2. $N \subseteq [N : a]$ and $N \subseteq [N : I]$

Lemma 2.5. Let P be a proper ideal of a semiring R. The following statements are equivalent.

1. P is a weakly prime ideal.
2. P is a weakly classical prime ideal.
3. $(P : I)$ is a weakly prime ideal, for each ideal I of R such that $I \not\subset P$.

5. $\sqrt{(P:I)}$ is a weakly prime ideal, for each ideal I of R such that $I \not\subset P$.

Proof. 1. $\Rightarrow$ 2. Suppose that P is a weakly primary ideal of R. We will show that P is a weakly classical primary ideal of R. Let $0 \neq abJ \subseteq P$, where $a, b \in R$ and J is an ideal of R such that $bJ \not\subset P$. Then, there exists $x \in bJ$ such that $x \notin P$. Since P is weakly primary ideal and $0 \neq ax \in P$, we conclude that $a^n \in P$, for some positive integer n. It follows that $a^n J \subseteq P$, for some positive integer n. Thus, P is a weakly classical primary ideal of R.

2. $\Rightarrow$ 3. Suppose that P is a weakly classical primary ideal of R. We will show that $(P : I)$ is a weakly primary ideal, for each ideal I of R such that $I \not\subset P$. Let $0 \neq ab \in (P : I)$, where $a, b \in R$. Then, $0 \neq abI \subseteq P$, so that $aI \subseteq P$ or $b^n I \subseteq P$, for some positive integer n. It follows that $a \in (P : I)$ or $b^n \in (P : I)$, for some positive integer n. Thus, $(P : I)$ is a weakly primary ideal of R.

3. $\Rightarrow$ 1. Suppose that $(P : I)$ is a weakly primary ideal, for each ideal I of R such that $I \not\subset P$. We will show that P is a weakly primary ideal of R. Take $I = R$ and so by 3, $P = (P : R)$ is a weakly primary ideal of R.

3. $\Rightarrow$ 5. is evident.

4. $\Rightarrow$ 5. Suppose that P is a weakly classical quasi-primary ideal of R. We will show that $\sqrt{(P:I)}$ is a weakly prime ideal, for each ideal I of R such that $I \not\subset P$. Let $0 \neq ab \in \sqrt{P:I}$, where $a, b \in R$. Then, $0 \neq (ab)^n \in (P : I)$, for some positive integer n, so that $0 \neq (ab)^n I \subseteq P$. Since P is a weakly classical quasi-primary ideal of R, there exists $k \in$ such that either $a^{nk} I \subseteq P$ or $b^{nk} I \subseteq P$, i.e., either $a \in \sqrt{P : I}$ or $b \in \sqrt{(P:I)}$. Thus, $\sqrt{(P:I)}$ is a weakly prime ideal of R.

5. $\Rightarrow$ 4. Suppose that $\sqrt{P : I}$ is a weakly prime ideal, for each ideal I of R such that $I \not\subset P$. We will show that P is a weakly classical quasi-primary ideal of R. Let $0 \neq abI \subseteq P$, where $a, b \in R$. Then, $0 \neq ab \in (P : I) \subseteq \sqrt{(P:I)}$. Since $\sqrt{(P:I)}$ is a weakly prime ideal of R, we have $\sqrt{(P:I)}$ is either R or a prime ideal of R, depending on whether $P \subseteq I$ or not, we conclude that $a \in \sqrt{P : I}$ or $b \in \sqrt{P : I}$, i.e., $a^n I \subseteq P$ or $b^n I \subseteq P$, for some positive integer n. Thus, P is a weakly classical quasi-primary ideal of R.

3. WEAKLY CLASSICAL PRIMARY AND CLASSICAL PRIMARY SUBSEMIMODULES

We start with the following theorem that gives a relation between weakly classical primary and classical primary subsemimodules in a semimodules over a semirings. Our starting points is the following lemma:

Lemma 3.1. Let N be a subsemimodule of semimodule M over a semiring R. Then, N is weakly classical primary if and only if for every $m \in M$ such that $m \notin N$, $(N : m)$ is a weakly primary ideal of R.

Proof. $\Rightarrow$ Suppose that N is a weakly classical primary subsemimodule of M. We will show that $(N : m)$ a weakly primary ideal, for each $m \in M$ such that $m \notin N$. Let $0 \neq ab \in (N : m)$, where $a, b \in R$. Then $0 \neq abm \in N$. By weakly classical primary subsemimodule, we have $am \in N$ or $b^n m \in N$, for some positive integer n. Thus $a \in (N : m)$ or $b^n \in (N : m)$, for some positive integer n. Hence $(N : m)$ a weakly primary ideal of R.

$\Leftarrow$ Suppose that $(N : m)$ is a weakly primary ideal of R, for each $m \in M$ such that $m \notin N$. We will show that N is weakly classical primary of M. Let $0 \neq abm \in N$, where $a, b \in R$. Then $0 \neq ab \in (N : m)$. Since $(N : m)$ is a weakly primary ideal of R, we have $a \in (N : m)$ or $b^n \in (N : m)$, for some positive integer n. It follows that $am \in N$ or $b^n m \in N$, for some positive integer n. By weakly classical primary subsemimodule, we have N is weakly classical primary of M.

Proposition 3.2. Let N be a k-subsemimodule of semimodule M over a semiring R. The following statements are equivalent.

1. N is a weakly classical primary subsemimodule.
2. For every subsemimodule K of M and $a, b \in R$. If $0 \neq abK \subseteq N$, then $aK \subseteq N$ or $b^n K \subseteq N$, for some positive integer n.

Proof 1. $\Rightarrow$ 2. Assume that N is a weakly classical primary subsemimodule of M. Let $a, b \in R$ and let K be a subsemimodule of M such that $0 \neq abK \subseteq N$. If $aK \not\subset N$ and $b^n K \not\subset N$, for all positive integer n, then there exist $x, y \in K$ such that $ax \notin N$ and $b^n y \notin N$ for all positive integer n. In this case from $0 \neq abx, aby \in abK \subseteq N$ we get $b^k x \in N$ and $ay \in N$ for some positive integer k. In this case it follows from

$0 \neq ab(x+y) \in abK \subseteq N$ that either $ax+ay = a(x+y) \in N$ or $b^k x + b^k y = b^k(x+y) \in N$ for some positive integer k. If $ax+ay \in N$, then $ax \in N$ since $ay \in N$ and N is a k-subsemimodule, a contradiction. If $b^k x + b^k y \in N$ we get a contradiction in a similar way.

2. $\Rightarrow$ 1. is evident

Proposition 3.3. Let N be a proper subsemimodule of semimodule M over a semiring R. If N is weakly classical primary subsemimodule of M, then $[N:c]$ is a weakly classical primary subsemimodule of M, where $c \in R$.

Proof. Let $0 \neq abm \in [N:c]$, where $a, b \in R$ and $m \in M$. By Definition 2.4, we have $0 \neq (ca)bm = c(abm) \in N$. Since $0 \neq (ca)bm \in N$ and N is weakly classical primary subsemimodule of M, we have $cam \in N$ or $b^n m \in N$, for some positive integer n. Then $am \in [N:c]$ or $b^n m \in N \subseteq [N:c]$, for some positive integer n. Hence $[N:c]$ is a weakly classical primary subsemimodule of M.

Proposition 3.4. Let R be a semiring, with identity, and let N be a proper subsemimodule of semimodule M over R. If $[N:c]$ is a classical primary subsemimodule of M, then N is weakly classical primary subsemimodule of M, where $c \in R$.

Proof. Let $0 \neq abm \in N$, where $a, b \in R$ and $m \in M$. Then $0 \neq 1bm = bm \in [N:a]$. Since $0 \neq 1bm \in [N:a]$ and $[N:a]$ is weakly classical primary subsemimodule of M, we have $m = 1m \in N \subseteq [N:a]$ or $b^n m \in N$, for some positive integer n. Thus $am \in N$ or $b^n m \in N$, for some positive integer n, and hence N is weakly classical primary of M.

Theorem 3.5. Let N be a k-subsemimodule of semimodule M over a semiring R. Then, N is weakly classical primary if and only if for every subsemimodule K of M such that $K \not\subset N$, $(N:K)$ is a weakly primary ideal of R.

Proof. $\Rightarrow$ Suppose that N is a weakly classical primary subsemimodule of M. We will show that $(N:K)$ a weakly primary ideal, for each subsemimodule K of M such that $K \not\subset N$. Let $0 \neq ab \in (N:K)$, where $a, b \in R$. Then $0 \neq abK \subseteq N$. By Proposition 3.2, we have $aK \subseteq N$ or $b^n K \subseteq N$, for some positive integer n. Thus $a \in (N:K)$ or $b^n \in (N:K)$, for some positive integer n. Hence $(N:K)$ a weakly primary ideal of R.

$\Leftarrow$ Suppose that $(N:K)$ is a weakly primary ideal of R, for each subsemimodule K of M such that $K \not\subset N$. We will show that N is weakly classical primary of M. Let $0 \neq abK \subseteq N$, where $a, b \in R$. Then $0 \neq ab \in (N:K)$. Since $(N:K)$ is a weakly primary ideal of R, we have $a \in (N:K)$ or $b^n \in (N:K)$, for some positive integer n. It follows that $aK \subseteq N$ or $b^n K \subseteq N$, for some positive integer n. By Proposition 3.2, we have N is weakly classical primary of M.

Theorem 3.6. Let M be a semimodule over a commutative semiring R. If N is a weakly classical primary subsemimodule of M, then $\sqrt{[N:ab]} = \sqrt{[N:a]} \cup \sqrt{[N:b]} \cup \sqrt{[0:ab]}$, for some positive integer n.

Proof. Let N be a weakly classical primary subsemimodule of M. We will show that $\sqrt{[N:ab]} \subseteq \sqrt{[N:a]} \cup \sqrt{[N:b]} \cup \sqrt{[0:ab]}$. Let $m \in \sqrt{[N:ab]}$. Then $a^n b^n m = (ab)^n m \in N$, for some positive integer n. If $0 \neq a^n b^n m$ and N is a weakly classical primary subsemimodule of M, then $a^n m \in N$ or $b^{nk} m = (b^n)^k m \in N$, for some positive integer k. Thus $m \in \sqrt{[N:a]}$ or $m \in \sqrt{[N:b]}$ and hence $\sqrt{[N:ab]} \subseteq \sqrt{[N:a]} \cup \sqrt{[N:b]} \cup [N:0]$. If $0 = a^n b^n m = (ab)^n m$, then $m \in [0:ab]$. Next to show that

$$\sqrt{[N:a]} \cup \sqrt{[N:b]} \cup \sqrt{[0:ab]} \subseteq \sqrt{[N:ab]}.$$

Let $m \in \sqrt{[N:a]} \cup \sqrt{[N:b]} \cup \sqrt{[0:ab]}$. Then $m \in \sqrt{[N:a]}$, $m \in \sqrt{[N:b]}$ or $m \in \sqrt{[0:ab]}$ so that $a^n m$, $b^k m \in N$ or $(ab)^l m = 0$, for some $n, k, l \in$. Since N is a subsemimodule of M, we have

$$a^n b^n m, a^k b^k m \in N$$

or

$$(ab)^l m = 0 \in N.$$

So that $m \in \sqrt{[N:ab]}$ and hence $\sqrt{[N:ab]} = \sqrt{[N:a]} \cup \sqrt{[N:b]} \cup \sqrt{[0:ab]}$.

Theorem 3.5. Let M be a semimodule over a commutative semiring R and let N be a k-subsemimodule of M. If N is a weakly classical primary that is not classical primary. Then $\sqrt{(N:M)}N = \sqrt{0}$.

Proof. Let M be a semimodule over a commutative semiring R and let N be a k-subsemimodule of M. First, we prove that $\sqrt{(N:M)}N = \sqrt{0}$. Suppose that $(N:M)_2 N \neq 0$ to show that N is classical primary subsemimodule. Let $abm \in N$, where $a, b \in R$, $m \in M$. If $abm \neq 0$, then either

$$am \in N \text{ or } b^n m \in N$$

for some positive integer n, since N is weakly classical primary subsemimodule. So suppose that $abm = 0$. If $aRN \neq 0$ or $RbN \neq 0$, then there is an element m' of N such that $arm' \neq 0$, so that

$$0 \neq abrm' = 0 + abm' = abm + abrm' = ab(m + rm') \in N,$$

and hence P weakly classical primary subsemimodule gives either

$$am + arm' = a(m + rm') \in N$$

or

$$b^n m + b^n rm' = b^n (m + rm') \in N,$$

for some positive integer n. Thus $am \in N$ or $b^n m \in N$ for some positive integer n. So we can assume that $aRN = 0$ and $RbN = 0$, If $(N:M)^2 m \neq 0$, then there exists $r, s \in (N:M)$ such that $rsm \neq 0$. So

$$0 \neq rsm = abm + asm + rbm + rsm = (a + r)(b + s)m \in N$$

implies either $(a + r)m \in N$ or $(b + s)^n m \in N$, for some positive integer n and hence either $am \in N$ or $b^n m \in N$. So suppose $(N:M)^2 m = 0$. Since $(N:M)^2 N \neq 0$, there exists $c, d \in (N:M)$ and $m' \in N$ such that $cdm' \neq 0$. Then

$$\begin{aligned}(a + c)(b + d)(m + m') &= [(a + c)b + (a + c)d](m + m') \\ &= [ab + cb + ad + cd](m + m') \\ &= (ab + cb + ad + cd)m + (ab + cb + ad + cd)m' \\ &= abm + cbm + adm + cdm + abm' + cbm' + adm' + cdm' \\ &= cdm' \neq 0\end{aligned}$$

so either $(a + c)(m + m') \in N$ or $(b + d)^n (m + m') \in N$, for some positive integer n and hence either $am \in N$ or $b^n m \in N$. Thus N is classical primary subsemimodule. Clearly, $\sqrt{0} \subseteq \sqrt{(N:M)}N$. As $(N:M)^2 N = 0$ we get $\sqrt{(N:M)}N \subseteq \sqrt{0}$. hence $\sqrt{(N:M)}N = \sqrt{0}$, as required.

Corollary 3.6. Let M be a semimodule over a commutative semiring R and let N be a k-subsemimodule of M. If $\sqrt{(N:M)}N \neq \sqrt{0}$, then N is classical primary if and only if N is weakly classical primary.

Proof. This follows from Theorem 3.5.

Theorem 3.7. Let M be a semimodule over a commutative semiring R and let N be a subsemimodule of M. If $[N : ab] = [N : a] \cup \sqrt{[N : b]}$ or $[N : ab] = [0 : ab]$, then N is weakly classical primary subsemimodule of M.

Proof. Let M be a semimodule over a commutative semiring R and let N be a subsemimodule of M. Suppose that Let $0 \neq abm \in N$, where $m \in M$, $a, b \in R$. Then

$$m \in [N : ab] = [N : a] \cup \sqrt{[N : b]}$$

by **Definition 2.4** hence $m \in P$ since $abm \neq 0$, as required.

Lemma 3.8. Let $M = M_1 \times M_2$, where each M_i is a semimodule over a commutative semiring R_i. Then the following hold:

1. If N is a subsemimodule of M_1, then $\sqrt{N \times M_2} = \sqrt{N} \times M_2$.
2. If N is a subsemimodule of M_2, then $\sqrt{M_1 \times N} = M_1 \times \sqrt{N}$.

Proof. The proof is straightforward.

Theorem 3.9. Let $M = M_1 \times M_2$, where each M_i is a semimodule over a commutative semiring R_i. Then N is a weakly classical primary (classical primary) subsemimodule of M_1 if and only if $N \times M_2$ is a weakly classical primary (classical primary) subsemimodule of M.

Proof. $\Rightarrow$ Suppose that $M = M_1 \times M_2$, where each M_i is a semimodule over a commutative semiring R_i and N is a weakly classicaly primary subsemimodule of M_1. Let

$$0 \neq (a_1 b_1 m_1, a_2 b_2 m_2) = (a_1, a_2)(b_1, b_2)(m_1, m_2) \in N \times M_2,$$

where $(a_1, a_2), (b_1, b_2) \in R$, $m_i \in M_i$ so either $a_1 m_1 \in N$ or $a_2 m_2 \in \sqrt{N}$ since N is weakly classical primary. It follows that either

$$(a_1, b_1)(m_1, m_2) = (a_1 m_1, b_1 m_2) \in N \times M_2$$

or

$$(a_2, b_2)(m_1, m_2) = (a_2 m_2, b_2 m_2) \in \sqrt{N \times M_2} = \sqrt{N} \times M_2$$

By Definition of weakly classical primary subsemimodule, we have $N \times M_2$ is a weakly classical primary subsemimodule of M.

$\Leftarrow$ Suppose that $M = M_1 \times M_2$, where each M_i is a semimodule over a commutative semiring R_i and $N \times M_2$ is a weakly classical primary subsemimodule of M. Let $0 \neq a_1 b_1 m_1 \in N$, where $a_1, b_1 \in R_1$, $m_1 \in M_1$ so that

$$0 \neq (a_1, a_2)(b_1, b_2)(m_1, m_2) = (a_1 b_1 m_1, a_2 b_2 m_2) \in N \times M_2,$$

where $a_2, b_2 \in R$, $m_2 \in M_2$. Since $N \times M_2$ is weakly classical primary, we have

$$(a_1 m_1, a_2 m_2) = (a_1, a_2)(m_1, m_2) \in N \times M_2$$

or

$$(b_1 m_2, b_2 m_2) = (b_1, b_2)(m_1, m_2) \in \sqrt{N \times M_2} = \sqrt{N} \times M_2$$

It follows that either $a_1 m_1 \in N$ or $b_1 m_1 \in \sqrt{N}$. By Definition of weakly classical primary subsemimodule, we have N is a weakly classical primary subsemimodule of M_1.

Corollary 3.10. Let $M = M_1 \times M_2$, where each M_i is a semimodule over a commutative semiring R_i. Then N is a weakly classical primary (classical primary) subsemimodule of M_2 if and only if $M_1 \times N$ is a weakly classical primary (classical primary) subsemimodule of M.

Proof. This follows from Theorem 3.9.

Corollary 3.11. Let $M = \prod_{i=1}^{n} M_i$, where each M_i is a semimodule over a commutative semiring R_i. Then N_j is a weakly classical primary (classical primary) subsemimodule of M_j if and only if $M_1 \times M_2 \times \cdots \times M_{j-1} \times N_j \times M_{j+1} \times \cdots \times M_n$ is a weakly classical primary (classical primary) subsemimodule of M.

Proof. This follows from Theorem 3.9 and Corollary 3.10.

Theorem 3.12. Let $M = M_1 \times M_2$, where each M_i is a semimodule over a commutative semiring with identity R_i. If N is a weakly classical primary (classical primary) subsemimodule of M, then either $N = 0$ or N is a classical primary subsemimodule of M.

Proof. Let $M = M_1 \times M_2$, where each M_i is a semimodule over a commutative semiring R_i and let $N = N_1 \times N_2$ be a weakly classical primary subsemimodule of M. We can assume that $N \neq 0$. So there is an element (m_1, m_2) of N, with $(m_1, m_2) \neq (0, 0)$. If $m_2 \neq 0$, then

$$(0, 0) \neq (abm_1, m_2) = (a, 1)(b, 1)(m_1, m_2) \in N,$$

where $a, b \in R_1$, gives either

$$(am_1, m_2) = (a, 1)(m_1, m_2) \in N$$

or

$$(bm_1, m_2) = (b, 1)(m_1, m_2) \in \sqrt{N},$$

If $(am_1, m_2) \in N$, then $N = N_1 \times M_2$. We will show that N_1 is classical primary; hence N is weakly classical primary by Theorem 3.9. Let $abm_1 \in N_1$, where $a, b \in R_1$, $m_1 \in M$. Then

$$(0, 0) \neq (abm_1, m_2) = (a, 1)(b, 1)(m_1, m_2) \in N,$$

so either

$$(am_1, m_2) = (a, 1)(m_1, m_2) \in N$$

or

$$(bm_1, m_2) = (b, 1)(m_1, m_2) \in \sqrt{N} = \sqrt{N_1 \times M_2} = \sqrt{N_1} \times M_2$$

and hence either $am_1 \in N$ or $bm_1 \in \sqrt{N_1}$. By a similar argument, $N = N_1 \times M_2$ is classical primary.

Proposition 3.13. Let M be a semimodule over a commutative semiring R and let $A \subseteq N$ be a proper subsemimodule of M. Then the following hold:

1. If N is weakly classical primary (classical primary), then N/A is weakly classical primary (classical primary).
2. If A and N/A are weakly classical primary (classical primary), then N is weakly classical primary (classical primary).

Proof. 1. Let $0 \neq ab(m + A) = abm + A \in N/A$, where $a, b \in R$, $m \in M$ so that $abm \in N$. If $abm = 0 \in A$, then

$$ab(m + A) = abm + A = 0,$$

a contradiction. So if N is weakly classical primary, then either $am \in N$ or $bm \in \sqrt{N}$, hence either

$$a(m + A) = am + A \in N/A$$

or

$$b(m + A) = bm + A \in \sqrt{N/A},$$

as required.

2. Let $0 \neq abm \in N$, where a, $b \in R$, $m \in M$, so $ab(m+A) = abm + A \in N/A$. For $abm \in A$, if A is weakly classical primary, then either

$$am \in A \subseteq P \text{ or } bm \in \sqrt{A} \subseteq \sqrt{N}.$$

So we may assume that $abm \notin A$. Then either $am + A = a(m+A) \in N/A$ or $bm + A = b(m+A) \in \sqrt{N/A}$. It follows that either $am \in N$ or $bm \in \sqrt{N}$ as needed.

Theorem 3.14. Let M be a semimodule over a commutative semiring R and let N, K be weakly classical primary subsemimodules of M that are not classical primary. Then $N+K$ is a weakly classical primary subsemimodule of M.

Proof. Since $(N+K)/K \approx K/(N \cap K)$, we get that $(N+K)/K$ is weakly classical primary by Proposition 3.13 (1). Now the assertion follows from Proposition 3.13 (2).

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